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Elliptic Root System and Elliptic Artin Group

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1 Introduction

The concepts of elliptic root system, elliptic Dynkin diagram and elliptic Weyl group were introduced by K. Saito to describe the Milnor lattices and the flat structures of semi-universal deformations for simply elliptic singularities [10][11][12][13].

Furthermore, in[15], K.Saito and T.Takebayashi studied generators and relations of elliptic Weyl groups in terms of elliptic Dynkin diagrams (This presentation of elliptic Weyl group is a generalization of Coxeter system. See Theorem 2.1). In the paper, they also proposed the following problems: find generators and relations of "elliptic Lie algebras", "elliptic Hecke algebras" and elliptic Artin groups (the fundamental groups of the complements of the discriminant for simply elliptic singularities) in terms of the elliptic Dynkin diagrams.

In [14], applying Borcherd's cnstruction of vertex algebras[2][[3], K.Saito and D.Yoshii constructed the elliptic Lie algebras (which are isomorphic to the toroidal algebras [8]) and described generators and relations of them for homogeous elliptic Dynkin diagrams (This presentation is a generalization of Serre relations attached to the elliptic Dynkin diagrams. cf. [1],[17]).

In [18], H. van der Lek has given a presentation of the elliptic Artin groups (which he calls the extended Artin groups) using affine Dynkin diagrams. The aim of this note is to give another presentation of elliptic Artin groups in terms of elliptic Dynkin diagrams. In our presentations, the numbers of generators and relations are less than his ones. Moreover, as a by-product, we shall define elliptic Hecke algebras (which are subalgebras of Cherednik's double affine Hecke algebras [4],[5],[6]) and construct finite dimensional irreducible representations of them.

Here,we briefly explain H. van der Lek's description of the elliptic Artin groups. Let $C = (c_{i,j})_{0\leq i,j\leq l}$ be an affine Cartan matrix and $M = (m_{i,j})_{0\leq i,j\leq l}$ be the Coxeter matrix determined by $c_{i,j}$ as follows:

$$m_{i,j} = 2, 3, 4, 6, \infty \quad if \quad c_{i,j}c_{j,i} = 0, 1, 2, 3, \geq 4, \quad respectively.$$

Theorem 1.1 (H. van der Lek[18]) *Let R_a be an affine root system and $C(R_a)$ the affine Cartan Matrix of R_a. The elliptic Artin group $A(R_a)$ associated with R_a is generated by $\{s_0, s_1, \cdots, s_l, t_0, t_1, \cdots, t_l\}$ which satisfy the following relations:*

(A.1) $\quad s_is_js_i\cdots = s_js_is_j\cdots \quad$ *each side $m_{i,j}$ factors if $i \neq j$*

(A.2) $\quad t_it_j = t_jt_i$

(A.3) $\quad s_i t_j t_i^r = t_j t_i^r s_i \qquad 2r = -c_{j,i}$

(A.4) $\quad s_i t_j t_i^r s_i = t_j t_i^{r+1} \qquad 2r+1 = -c_{j,i}$

In this note, for simplicity, we shall only treat with elliptic Dynkin diagrams obtained by adding one vertex to affine Dynkin diagrams (see the appendix) and, for brevity, call them elliptic Dynkin diagrams hereafter.

2 Elliptic root systems and elliptic Weyl groups

We briefly explain elliptic root systems and elliptic Weyl groups following [12]. Let F be a vector space over $\mathbf{R}$ with symmetric bilinear form $(\cdot,\cdot)$ of signature (l_+, l_0, l_-), where l_+ (resp. l_-) is the dimension of a maximal positive (resp. negative) definite subspace of F and l_0 is the dimension of the radical of $(\cdot,\cdot)$. For $\alpha \in F$ such that $(\alpha,\alpha) \neq 0$, we define

$$\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha,$$

$$w_\alpha(u) = u - (u,\alpha^\vee)\alpha \qquad for \quad any \quad u \in F.$$

Definition 2.1 *A subset $R \subset F$ is called an elliptic root system, if the following conditions are satisfied:*

(R.1) $\quad (l_+, l_0, l_-) = (l, 2, 0).$

(R.2) *Let $Q(R)$ be the $\mathbf{Z}$-submodule of F generated by R. Then, $Q(R)\otimes_{\mathbf{Z}}\mathbf{R} = F$.*

(R.3) *For any $\alpha \in R$, $(\alpha,\alpha) \neq 0$.*

(R.4) $\quad w_\alpha(R) = R.$

(R.5) $\quad (\alpha, \beta^\vee) \in \mathbf{Z}$ *for any $\alpha, \beta \in R$.*

(R.6) *If $\quad R = R_1 \cup R_2$, then $R_1 = \phi \quad or \quad R_2 = \phi$.*

We call the Weyl group $W(R)$ associated with R *elliptic Weyl group* of R. Also K. Saito defined an elliptic Dynkin diagram for an elliptic root system as follows: Let G be a 1-dimensional subspace of $rad(\cdot,\cdot)$ which is defined over $\mathbf{Q}$. Then, the quotient of R by G is an affine root system R_a. We fix a generator a of the lattice $G \cap Rad((,))$. Note that the generator a is unique up to a choice of sign. We call (R, G) the marked elliptic root system. For any $\alpha \in R_a$, put

$$k(\alpha) = inf\{k \in N | \alpha + k \cdot a \in R\}$$

and

$$\alpha^* = \alpha + k(\alpha) \cdot a.$$

$k(\alpha)$ is called the counting of α. Then we have the following proposition:

Proposition 2.1 (K.Saito [12]) *Let (R, G) be a marked elliptic root system. Then we have*

$$R = \{\alpha + m \cdot k(\alpha) \cdot a | \alpha \in R_a,\ m \in Z\}.$$

Let $\Gamma_a = \{\alpha_0, \alpha_1, \cdots, \alpha_l\}$ be a basis of R_a such that
$\{\alpha_1, \cdots, \alpha_l\}$: simple roots of finite root system
$\alpha_0 = b - \Sigma_{i=1}^{l} n_i \alpha_i$
b : imaginary root of R_a. The set of exponents of (R, G) is defined by the union of 0 and

$$m_\alpha = \frac{(\alpha, \alpha)_R}{2 \cdot k(\alpha)} \cdot n_\alpha \qquad for\ \alpha \in \Gamma_a$$

where $(\cdot, \cdot)_R$ is a constant multiple of $(\cdot, \cdot)$ normalized such that $inf\{(\alpha, \alpha)_R | \alpha \in R\}$ is equal to 2. Set

$$\Gamma_{a,max} = \{\alpha \in \Gamma_a | m_\alpha = max\{m_\beta | \beta \in \Gamma\}\}$$

and

$$\Gamma_{a,max}^* = \{\alpha^* | \alpha \in \Gamma_{a,max}\}.$$

Remark 2.1 *In this note, we assume that the number of the verteces of $\Gamma_{a,max}$ is equal to 1.*

Definition 2.2 *The elliptic Dynkin diagram $\Gamma(R, G)$ of the marked elliptic root system (R, G) is a finite graph generated by the set of verteces $\Gamma(R, G) = \Gamma_a \cup \Gamma_{a,max}^*$ and bonded by the following conditions: for $\alpha,\ \beta \in \Gamma(R, G)$,*

(1) $(\alpha, \beta^\vee) = 0$ $\qquad \alpha \bigcirc \quad \bigcirc \beta$

(2) $(\alpha, \beta^\vee) = (\alpha^\vee, \beta) = -1$ $\qquad \alpha \bigcirc\!\!-\!\!-\!\!-\!\!\bigcirc \beta$

(3) $(\alpha, \beta^\vee) = -t, (\alpha, \beta^\vee) = -1 \quad for\ t = 2, 3, 4$ $\qquad \alpha \bigcirc\!\!\underset{t}{-\!\!-\!\!-}\!\!\bigcirc \beta$

(4) $(\alpha, \beta^\vee) = (\alpha^\vee, \beta) = 2$ $\qquad \alpha \bigcirc\!\!=\!\!=\!\!=\!\!\bigcirc \beta$

Let us define the equality

$$m(R, G) = \frac{max\{m_\alpha | \alpha \in \Gamma(R, G)\}}{gcd\{m_\alpha | \alpha \in \Gamma_a\}}$$

This number $m(R, G)$ plays the role of the Coxter nummber for the elliptic root system [11],[12],[13],[15].

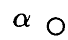
Using elliptic Dynkin diagrams, K. Saito and T. Takebayashi [15] gave presentations of elliptic Weyl groups as follows:

Theorem 2.1 (K. Saito and T. Takebayashi [15]) *Let (R, G) be a marked elliptic root system and $\tilde{W}(R, G)$ the group defined by the following generators and relations:*

generators : $\quad r_\alpha \qquad \alpha \in \Gamma(R, G)$

relations:

(W.0) $\quad r_\alpha{}^2 = 1$

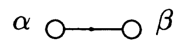


(W.1.0) $\quad (r_\alpha r_\beta)^2 = 1$

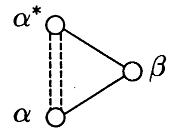
(W.1.1) $\quad (r_\alpha r_\beta)^3 = 1$

(W.1.2) $\quad (r_\alpha r_\beta)^4 = 1$

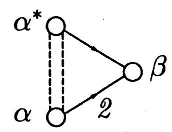
(W.1.3) $\quad (r_\alpha r_\beta)^6 = 1$



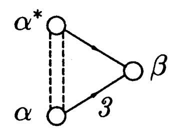
(W.2.1) $\quad (r_\alpha r_\beta r_{\alpha^*} r_\beta)^3 = 1$



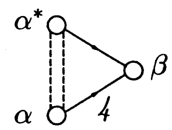
(W.2.2) $\quad (r_\alpha r_\beta r_{\alpha^*} r_\beta)^2 = 1$



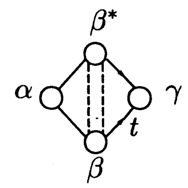
(W.2.3) $\quad (r_\alpha r_\beta r_{\alpha^*} r_\beta)^3 = 1$ *and* $(r_\alpha r_\beta r_{\alpha^*} r_\beta r_\alpha r_\beta)^2 = 1$



(W.2.3) $\quad (r_\alpha r_{\alpha^*} r_\beta)^2 = (r_{\alpha^*} r_\beta r_\alpha)^2 = (r_\beta r_\alpha r_{\alpha^*})^2$



(W.3) $\quad (r_\alpha r_\beta r_\alpha r_{\beta^*} r_\gamma r_{\beta^*})^2 = 1$ *and* $(r_\alpha r_\beta^* r_\alpha r_\beta r_\gamma r_\beta)^2 = 1$
for t=1,2,3



where the two relations are equivalent in the case of $t = 1$.
Define

$$\tilde{c}(R,G) \quad = \quad \prod_{i \in \{0,1,\cdots,l\} \setminus J} r_{\alpha i} \cdot \prod_{j \in J} r_{\alpha j} r^*_{\alpha j}.$$

Then,the power $\tilde{c}(R,G)^{m(R,G)}$ *is a center of* $\tilde{W}(R,G)$ *and one has*

$$\tilde{W}(R,G) / < \tilde{c}(R,G)^{m(R,G)} > \quad \cong \quad W(R).$$

Here the relations $(W.1.0) \sim (W.1.3)$ are well known as Coxeter relations, and the relations $(W.2.1) \sim (W.3.t)$ are newly introduced relations due to the double bonds in the diagram. Let us call them *elliptic Coxeter relations* and the group $\tilde{W}(R,G)$ *hyperbolic elliptic Weyl group* of $W(R)$ (see [12,[15]).

Remark 2.2 *In this note, we have assumed that the number of the vertces of* $\Gamma_{a,max}$ *is equal to 1. In [15],K.Saito and T.Takebayashi treated more general elliptic root systems.*

3 Twisted Picard-Lefschetz formula

The Coxeter relations of elliptic Weyl groups were obtained by studying the monodromy representations of simply elliptic singularities. To find generators and their relations of elliptic Artin groups attached to elliptic Dynkin diagrams, we want to construct a certain kind of deformation of the monodromy representations of simply elliptic singularities. Fortunately, A. B. Givental[7] has already studied q-deformation of monodromy representations of isolated hypersurface singularities using the so called twisted Picard-Lefschetz formula (see also F. Pham[9] and I.Shimada[16]).

First we begin with explaining the classical Picard-Lefschetz formula.
Let $f : (\mathbf{C}^3, 0) \longrightarrow (\mathbf{C}, 0)$ be a polymonial mapping such that $f^{-1}(0)$ has a simply elliptic singularity. Namely,

$$\left\{ \begin{array}{llcccccc} E_6^{(1,1)}: & f(x,y,z) & = & x^3 & + & y^3 & + & z^3 \\ E_7^{(1,1)}: & f(x,y,z) & = & x^4 & + & y^4 & + & z^2 \\ E_8^{(1,1)}: & f(x,y,z) & = & x^6 & + & y^3 & + & z^2. \end{array} \right.$$

Since a simply elliptic singilarity has a semi-universal deformation, there exists the following commutative diagram which is called a Hamiltonian system in[10]:

$$\begin{tikzcd} X = \mathbf{C}^3 \times \mathbf{C}^{\mu-1} \arrow[r, hook, bend left, "F_1"] \arrow[d, "pr_2"'] \arrow[dr, "\phi"] & \mathbf{C}^3 \times \mathbf{C}^{\mu-1} \times \mathbf{C} = Z \arrow[l, "\tilde{\pi}"] \arrow[d, "pr_1"] \\ T = \mathbf{C}^{\mu-1} & \mathbf{C}^{\mu-1} \times \mathbf{C} = S \arrow[l, "\pi"] \end{tikzcd}$$

where

$$\left\{ \begin{array}{c} \tilde{\pi}, \pi, pr_1, pr_2 : \text{natural projections}, \\ F_1(x,y,z,t_1,\cdots,t_{\mu-1}) = (x,y,z,t_1,\cdots,t_{\mu-1}, \hat{F}_1(x,y,z,t_1,\cdots,t_{\mu-1})), \\ \hat{F}_1(x,y,z,t_1,\cdots,t_{\mu-1}) = f(x,y,z) + \sum_{j=1}^{\mu-1} t_j\phi_j \end{array} \right.$$

and

$$\phi = pr_1 \circ F_1 : \quad \text{semi-universal deformation of } f.$$

Here $\mu = l+2$ and $\{\phi_j\}_{j=1}^{\mu}$ is a C-basis of the Jacobi ring $\mathbf{C}[x,y,z]/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ of f such that $deg(\phi_{j+1}) \leq deg(\phi_j)$.

Let C_ϕ be the critical set of ϕ and D_ϕ the discriminant of ϕ. The discriminant D_ϕ is a reduced irreducible hypersurface in S. Let $t' \in T = \mathbf{C}^{\mu-1}$ be a point which is not contained in the image of the ramification locus of $\pi|_{D_\phi}$. Set

$$L_{t'} = \{t'\} \times \mathbf{C} \subset \mathbf{C}^{\mu-1} \times \mathbf{C} = S.$$

By choice of t', there are exactly μ intersection points of $L_{t'}$ with the discriminant D_ϕ. We denote these points $p_1, \cdots, p_\mu$. A fibre $X_{p_i} = \phi^{-1}(p_i)$ has a singularity which is the ordinary double point. Let $p_0 \in L_{t'} \backslash \{p_1, \cdots, p_\mu\}$. Then the fibre $X_{p_0} = \phi^{-1}(p_0)$ is a 2-dimensional manifold and homotopically isomorphic to a bouquet of μ copies of sphere S^2 . Hence, the only non-trivial homology group of X_{p_0} is the group $H_2(X_{p_0}, \mathbf{Z})$ which is a free Z-module of rank μ.

The intersection numbers of cycles define a symmetric bilinear form (,) on this module with signature (l, 2, 0) ($\mu = l+2$).

Next, we shall explain a relation between elliptic Dynkin diagram and vanishing cycles. Choose a simple arc l_i in $L_{t'}$ from p_0 to p_i not passing through other p_j. Then

$$X_{p_0} \subset \phi^{-1}(l_i) \longrightarrow X_{p_i} : \quad \text{contraction}$$

induces the mapping

$$c_i : H_2(X_{p_0}, \mathbf{Z}) \longrightarrow H_2(X_{p_i}, \mathbf{Z}).$$

The kernel of this mapping is a $\mathbf{Z}$-submodule of $H_2(X_{p_0}, \mathbf{Z})$ of rank 1. Denote a generator of Kernel(c_i) by e_i, i.e.

$$\text{Kernel}(c_i) = \mathbf{Z}e_i.$$

It can be shown that if $l_1, \cdots, l_\mu$ are chosen in such a way that l_i and l_j intersect only at p_0 for $i \neq j$, then $\{e_1, \cdots, e_\mu\}$ is a free $\mathbf{Z}$-basis of $H_2(X_{p_0}, \mathbf{Z})$ and $H_2(X_{p_0}, \mathbf{Z}) = Q(R)$. Furthermore the intersection matrix with respect to this basis determines the elliptic Dynkin diagram (see [10],[11]).

Now, we explain the classical Picard-Lefschetz formula. To each path l_i, we associate an element $\gamma_i \in \pi_1(L_{t'}, p_0)$ by going along l_i from p_0 to a point near p_i, then turning counterclockwise in a small circle around p_i and then returning to p_0 along l_i. Then $\{\gamma_1, \cdots, \gamma_\mu\}$ is a set of generators of $\pi_1(S \backslash D_\phi, p_0)$. The mapping

$$\phi : \phi^{-1}(S \backslash D_\phi) \longrightarrow S \backslash D_\phi$$

is the projection of a fibre bundle. Hence one gets a monodromy representation

$$\rho : \pi_1(S \backslash D_\phi, p_0) \longrightarrow Aut(H_2(X_{p_0}, \mathbf{Z})).$$

Finally, we can state the classical Picard-Lefschetz formula:

Classical Picard-Lefschetz formula (see [9])

$$\rho(\gamma_i)(\alpha) = \alpha - (\alpha, e_i)e_i \qquad \text{for any } \alpha \in H_2(X_{p_0}, \mathbf{Z})$$

$$(i = 1, \cdots, \mu = l + 2)$$

Now, according to A. B. Givental [7], we explain the twisted Picard-Lefschetz formula. Define $\hat{F} : Z \to \mathbf{C}$ by

$$\hat{F}(x, y, z, t_1, \cdots, t_\mu) = \hat{F}_1(x, y, z, t_1, \cdots, t_{\mu-1}) + t_\mu$$

and

$$\tilde{Z} = Z \backslash F^{-1}(0).$$

Since $\pi_1(\tilde{Z}) \cong \mathbf{Z}$, for a complex number $q \in \mathbf{C}^*$, we can define a representation

$$\pi_1(\tilde{Z}) \to Aut(\mathbf{C}) : \qquad 1 \mapsto q.$$

This representation induces a local system $\mathcal{L}_q$ on $\tilde{Z}$. Define $\tilde{Z}^r = pr_1^{-1}(S \backslash D_\phi) \cap \tilde{Z}$ and then $pr_1 : \tilde{Z}^r \to S \backslash D_\phi$ is a fibre bundle whose fibre is 3-dimensional complex manifold. For simplicity, we also denote by $\mathcal{L}_q$ the restriction of $\mathcal{L}_q$ to the fibre $\tilde{Z}^r(p_0) = pr_1^{-1}(p_0)$. Then we get a monodromy representation

$$\rho_q : \pi_1(S \backslash D_\phi) \longrightarrow Aut_{\mathbf{Z}[q,q^{-1}]}(H_3(\tilde{Z}^r(p_0), \mathcal{L}_q)).$$

This monodromy representation can be regarded as a q-deformation of the classical one.

Denote the restriction of the mapping $\hat{F}$ to $\tilde{Z}^r(p_0)$ by

$$\tilde{F}_{p_0} : \tilde{Z}^r(p_0) \longrightarrow \mathbf{C}^*.$$

By choice of p_0, $\tilde{F}_{p_0}$ has exactly μ critical values. We denote these points by $p_1', \cdots, p_\mu'$. Choose a simple arc γ_i' in $\mathbf{C}^*$ going from p_i to a point near the origin, then turning counterclockwise in a small circle around it, and finally returning to p_i' along the same way, and define a cycle $\delta_i \in H_3(\tilde{Z}^r(p_0), \mathcal{L}_q)$ by carrying the vanishing cycle e_i along γ_i'. Then we obtain the following:

Theorem 3.1 (A.B. Givental [7]) (1) $H_3(\tilde{Z}^r(p_0), \mathcal{L}_q) = \oplus_{j=1}^{\mu} \mathbf{Z}[q, q^{-1}]\delta_j$

(2) *Let V be an upper triangular matrix with diagonal elements 1 and (e_i, e_j) for $i < j$ and define a $\mu \times \mu$-matrix $I_q = qV +^t V$. Then one has*

$$\rho_q(\gamma_i)(\delta_j) = \delta_j - I_{q_{i,j}}\delta_i.$$

(3) $(\rho_q(\gamma_i) + q)(\rho_q(\gamma_i) - 1) = 0.$

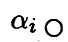
Applying this theorem to our problem, we obtain the following proposition:

Proposition 3.1 *Set*

$$g_i' = \rho_q(\gamma_i) \qquad (i = 1, \cdots, \mu = l + 2).$$

Then $g_1', \cdots, g_\mu'$ satisfy the following relations:

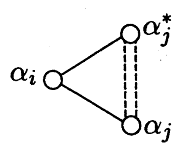
(1) $\quad (g_i' + q)(g_i' - 1) = 0$



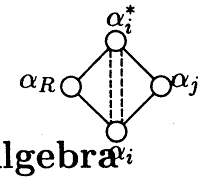
(2) $\quad g_i'g_j' = g_j'g_i'$

$\quad g_i'g_j'g_i' = g_j'g_i'g_j'$

(3) $\quad$ *Let $t_j' = g_j'g_j'^*$, then $g_i't_j'g_i't_j' = t_j'g_i't_j'g_i'$*



(4) $\quad$ *Let $t_j' = g_j't_i'g_j't_i'^{-1}$, then $g_k't_j' = t_j'g_k'$*



4 Elliptic Artin group and elliptic Hecke algebra

Motivated by Proposition 3.2, we define the following group:

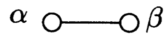
Definition 4.1 *Let (R, G) be a marked elliptic root system and $\Gamma(R, G)$ be its elliptic Dynkin diagram. Define a group $\tilde{A}(R, G)$ by the following generators and their relations:*

generators: g_α $\qquad$ $\alpha \in \Gamma(R, G)$

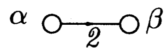
relations:

(E.1.0) $g_\alpha g_\beta = g_\beta g_\alpha$

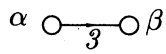
(E.1.1) $g_\alpha g_\beta g_\alpha = g_\beta g_\alpha g_\beta$



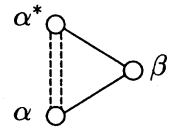
(E.1.2) $g_\alpha g_\beta g_\alpha g_\beta = g_\beta g_\alpha g_\beta g_\alpha$



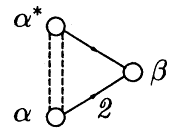
(E.1.3) $g_\alpha g_\beta g_\alpha g_\beta g_\alpha g_\beta = g_\beta g_\alpha g_\beta g_\alpha g_\beta g_\alpha$



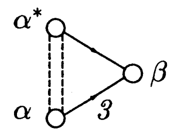
(E.2.1) *Let* $t_\alpha = g_\alpha g_{\alpha^*}$, *then* $g_\beta t_\alpha g_\beta t_\alpha = t_\alpha g_\beta t_\alpha g_\beta$



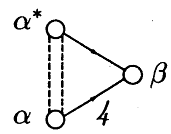
(E.2.2) $g_\beta t_\alpha g_\beta g_\alpha = g_\alpha g_\beta t_\alpha g_\beta$



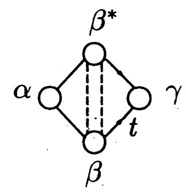
(E.2.3) $g_\beta t_\alpha g_\beta t_\alpha = t_\alpha g_\beta t_\alpha g_\beta$ *and* $g_\beta t_\alpha g_\beta g_\alpha = g_\alpha^* g_\beta t_\alpha g_\beta$



(E.2.4) $g_\beta t_\alpha g_\beta t_\alpha = t_\alpha g_\beta t_\alpha g_\beta = g_\alpha^* g_\beta t_\alpha g_\beta g_\alpha$



(E.3) $g_\alpha t_\gamma = t_\gamma g_\alpha$ *and* $g_\gamma t_\alpha = t_\alpha g_\gamma$
where $t_\gamma = g_\gamma t_\beta g_\gamma t_\beta^{-1}$ *and* $t_\alpha = g_\alpha t_\beta g_\alpha t_\beta^{-1}$
for t=1,2,3

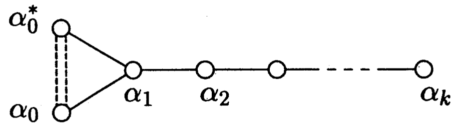


Here the relations (E.1.0) $\sim$ (E.1.3) are the same with (A.1) in Theorem 1.1 and the relations (E.3) $\sim$ (E.5) are newly introduced ones due to the double bonds in the diagram $\Gamma(R, G)$.

Now we consider the relation of $\tilde{A}(R, G)$ and the elliptic Artin group $A(R_a)$. To this purpose, we introduce $t_\alpha \in \tilde{A}(R, G)$ as follows: For $\alpha_0 \in \{\alpha | \alpha \in \Gamma_{a,max}\}$, $t_{\alpha 0}$ is already defined by

$$t_{\alpha 0} = g_{\alpha 0} g_{\alpha 0^*}.$$

If $\alpha_1, \cdots, \alpha_k \in \Gamma(R_a) \backslash \{\alpha | \alpha \in \Gamma_{a,max}\}$ are arranged the following position



then we define

$$t_{\alpha j+1} = g_{\alpha j+1} t_{\alpha j} g_{\alpha j+1} t_{\alpha j}^{-1}.$$

inductively. Then we obtain the following lemma:

Lemma 4.1 *Let $N(R,G)$ be a subgroup of $\tilde{A}(R,G)$ generated by $\{t_\alpha | \alpha \in \Gamma(R,G)\}$. Then one has*

(1) *$N(R,G)$ is a free abelian subgroup.*

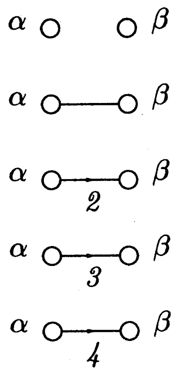
(2) $g_\alpha t_\beta = t_\beta g_\alpha$

$g_\alpha t_\beta g_\alpha = t_\alpha t_\beta$

$g_\beta t_\alpha t_\beta = t_\alpha t_\beta g_\beta$

$g_\beta t_\alpha t_\beta g_\beta = t_\alpha t_\beta^2$

$g_\beta t_\alpha t_\beta^2 = t_\alpha t_\beta^2 g_\beta$



(3) *Set* $c(R,G) = \prod_{\alpha \in \Gamma(R,G) \backslash \{\alpha_j | j \in J\}} g_\alpha \prod_{\alpha \in \{\alpha_j | j \in J\}} g_\alpha g_{\alpha^*}$, *then the power $c(R,G)^{m(R,G)}$ is a center of $\tilde{A}(R,G)$ and belongs to $N(R,G)$. Especially, $c(R,G)^{m(R,G)}$ is expressed by*

$$c(R,G)^{m(R,G)} = \prod_{\alpha \in \Gamma(R_a)} t_\alpha^{n_\alpha}$$

where n_α are the coefficients of the imaginary root of the affine root system R_a. Here $c(R,G)$ is called the Coxeter element of $\tilde{A}(R,G)$

By Theorem1.1 and this Lemma 4.1, we obtain the following theorem:

Theorem 4.1 *Let (R,G) be a marked elliptic root system and R_a the corresponding affine root system. Then the group $\tilde{A}(R,G)$ is isomorphic to the elliptic Artin group $A(R_a)$.*

Therefore, we obtain generators and their relations of an elliptic Artin group associated with an elliptic Dynkin diagram.

In [4],[5],[6], I.Cherednik defined the concept of double affine Hecke algebra and proved Macdonald's inner product conjecture. We shall define the elliptic Hecke algebra which can be proved to be a subalgebra of his algebra.

Definition 4.2 *Let (R, G) be a marked elliptic root system. For $q \in \mathbf{C}^*$, the elliptic Hecke algebra $H_q(R, G)$ associated with (R, G) is the quotient of the group algebra $\mathbf{C}(q)[\tilde{A}(R, G)]$ by the relations*

$$(E.0) \qquad (g_\alpha + q)(g_\alpha - 1) = 0 \qquad \text{for } \alpha \in \Gamma(R, G).$$

where $\mathbf{C}(q)$ is the quotient field of $\mathbf{C}[q, q^{-1}]$.

Remark 4.1 **(1)** *When $q = 1$, the relations (E.1.0) $\sim$ (E.3) are equivalent to the elliptic Coxeter relations (W.1.0) $\sim$ (W.3).*

(2) *Cherednik's double affine Hecke algebra contains two parameters. In the elliptic Hecke algebra, two parameters appear from the local system $\mathcal{L}_q$ and the power of the Coxeter element, $c(R, G)^{m(R,G)}$.*

Let $C(R, G)$ be the Cartan matrix corresponding to an elliptic Dynkin diagram $\Gamma(R, G)$ and T be the upper triangular matrix with diagonal elements 1 such that

$$C(R, G) = T +^t T.$$

Difine $\mu \times \mu$-matrix

$$C_q(R, G) = q \cdot T +^t T,$$

whrere $\mu =$ the number of vertces of $\Gamma(R, G)$.

Note that we have assume that the number of vertces of $\Gamma_{a,max}$ is equal to 1.

On the vector space $V(R, G) = \oplus_{\alpha \in \Gamma(R,G)} \mathbf{C}(q)\alpha$, for any $\alpha \in \Gamma(R, G)$, define the element A_α of $Aut(V(R, G))$ as follows : for any $\beta \in \Gamma(R, G)$,

$$A_\alpha(\beta) = \beta - C_q(R, G)_{\alpha,\beta} \cdot \alpha,$$

where $C_q(R, G)_{\alpha,\beta}$ is (α, β)-component of $C_q(R, G)$. Then we obtain the following proposition:

Proposition 4.1 *Let (R, G) be a marked elliptic root system such that the number of vertces of $\Gamma_{a,max}$ is equal to 1. Then one has*

(1)

$$\rho_q : \tilde{A}(R, G) \longrightarrow Aut(V(R, G))$$

$$\rho_q(g_\alpha) = A_\alpha$$

is a fine dimensional irreducible representation of $\tilde{A}(R, G)$ over $\mathbf{C}(q)$.

(2) *The above representation induces the follwing commutative diagram:*

$$\begin{tikzcd} \rho_q : \mathbf{C}(q)[\tilde{A}(R, G)] \arrow[r] \arrow[d] & Aut(V(R, G)) \\ H_q(R, G). \arrow[ur] & \end{tikzcd}$$

Especially, one obtains a finite dimensional irreducible representation of $H_q(R, G)$.

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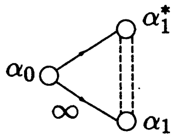
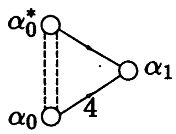
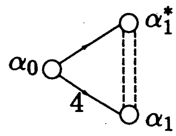
$A_1^{(1,1)*}$



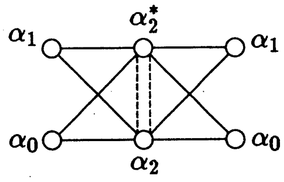
$BC_1^{(2,1)}$



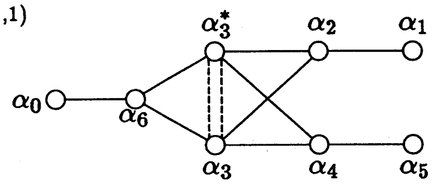
$BC_1^{(2,4)}$



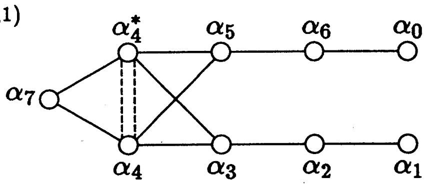
$D_4^{(1,1)}$



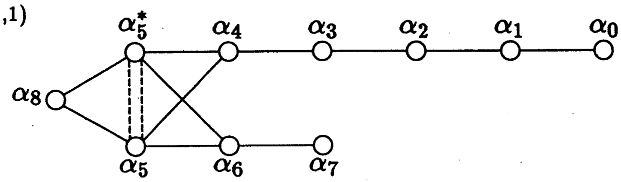
$E_6^{(1,1)}$



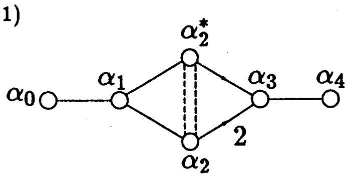
$E_7^{(1,1)}$



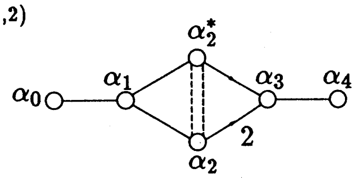
$E_8^{(1,1)}$



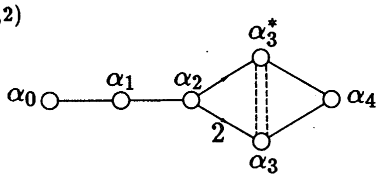
$F_4^{(1,1)}$



$F_4^{(2,2)}$



$F_4^{(1,2)}$



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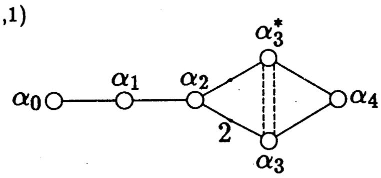
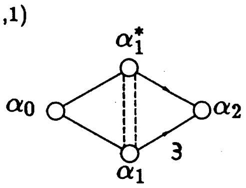
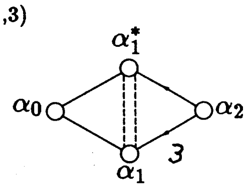


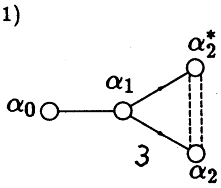
$G_2^{(1,1)}$



$G_2^{(1,3)}$



$G_2^{(3,1)}$



$G_2^{(3,3)}$